

AD-A123-913

RISK-EFFICIENT ESTIMATION OF THE MEAN EXPONENTIAL  
SURVIVAL TIME UNDER RAN..(U) MICHIGAN STATE UNIV EAST  
LANSING DEPT OF STATISTICS AND PROBA..

1/1

UNCLASSIFIED

J C GARDINER ET AL. JAN 83 N00014-79-C-0522 F/G 12/1

NL

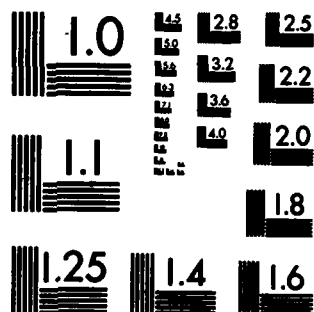
END

DATE

FILMED

B3

DTIC



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

ADA 123913

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS  
BEFORE COMPLETING FORM

Risk-efficient estimation of the mean exponential survival time under random censoring

Joseph C. Gardiner<sup>\*</sup>  
Michigan State University, East Lansing, Michigan 48824

V. Susarla<sup>†</sup>  
State University of New York, Binghamton, New York 13901

Abstract: The paper proposes a sequential estimator  $\hat{\theta}$  of the parameter  $\theta$  of an exponential distribution when the data is censored. Without any further conditions, it is shown that  $\hat{\theta}$  is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of  $\theta$  plus a linear cost function of the number of observations. In addition, it is shown that  $\hat{\theta}$  is asymptotically normal as the cost per observation goes to zero.

Key Words and Phrases: censored data, exponential distribution, and sequential estimation.

<sup>\*</sup>Research supported in part by the National Institutes of Health under grant 1R01-GM28030 and by the Office of Naval Research under contract N00014-79-C0522.

<sup>†</sup>Research supported in part by the National Institutes of Health under grant 1R01-GM28405.

# 1. Introduction.

In several longitudinal investigations the estimation of the mean survival time is of basic importance. This is usually based on the data gathered from a sample of  $n(\geq 1)$  identical units on test as in a clinical trial or life test. However, it is often the case in these survival studies that the lifetimes of the specimens are not completely observable due to random withdrawals and consequent loss to follow-up. Accordingly, for each survival time  $X$  we envisage a competing censoring time  $Y$ , but the only datum available is  $(Z, \delta)$  where  $Z = \min(X, Y)$  and  $\delta = 1$  whenever  $X \leq Y$  and  $\delta = 0$  otherwise. We shall assume  $X$  to be independent of  $Y$  and  $Z \geq 0$  almost surely (a.s.) for each  $\theta$ .

Suppose  $\{(Z_i, \delta_i): 1 \leq i \leq n\}$  is a random sample of size  $n$  with lifetimes  $X_i$  having common exponential survival function  $F(t) = \exp(-t/\theta)$ ,  $t > 0$  with  $\theta(> 0)$  unknown and the censoring times  $Y_i$  having common (unknown) survival function  $G$ . We consider the sequence of estimators  $\{\hat{\theta}_n\}$  of  $\theta$  given by

$$\hat{\theta}_n = \begin{cases} (\sum_{i=1}^n Z_i) / (\sum_{i=1}^n \delta_i) & \text{if } \sum_{i=1}^n \delta_i \geq 1 \\ 0 & \text{if } \sum_{i=1}^n \delta_i = 0 \end{cases} \quad [1.1]$$

$\hat{\theta}_n$  is easily seen to converge a.s. to  $\theta$  as  $n \rightarrow \infty$  for each  $\theta$ .

The loss incurred in estimation of  $\theta$  by  $\hat{\theta}_n$  is

$$L_n(a, c) = a(\hat{\theta}_n - \theta)^2 + cn \quad [1.2]$$

where  $a(> 0)$  is a known constant and  $c(> 0)$  is the cost per unit observation. With the definition [1.1] we can show that

$$E(\hat{\theta}_n - \theta)^2 = (a\theta^2/E\delta)n^{-1} + o(n^{-1}) \quad [1.3]$$

whence the risk associated with the estimation scheme becomes

$$R_n(a, c) = EL_n(a, c) = (a\theta^2/E\delta)n^{-1} + cn + o(n^{-1}).$$

Therefore, in order to minimize this risk (with respect to  $n$ ) we must

take a sample of size  $n_c^0$  with subsequent minimum risk  $R_c^0$  where (for small  $c$ )

$$R_c^0 \sim 2cn_c^0 \quad \text{and} \quad n_c^0 \sim (a\theta^2/cE\delta)^{1/2}. \quad [1.4]$$

Since  $\theta$  is the unknown parameter which we wish to estimate, this minimum risk  $R_c^0$  and optimal sample size  $n_c^0$  are also unknown and thus we are led naturally to explore a sequential scheme for estimating  $\theta$ . In this article we propose one such procedure in which the sample size is given by a random stopping number  $N(c)$  and the associated risk  $R_c^* = EL_{N(c)}$  satisfies  $R_c^*/R_c^0 \rightarrow 1$  as  $c \rightarrow 0$ , that is if our scheme is asymptotically risk-efficient.

Sequential procedures analogous to the one outlined here have been considered, in the absence of censoring, by several researchers beginning with the pioneering work of Robbins (1959) for the estimation of the mean of a normal population. This was later extended by Starr (1966) and Starr and Woodroffe (1969). The study of sequential point estimation of the exponential mean is taken up in Starr and Woodroffe (1972). Gardiner and Susarla (1982) make the first examination of the problem of sequential estimation of the mean survival time in the presence of censoring when both the underlying survival time and censoring distributions are unspecified. When censoring is absent, similar procedures for estimation of functionals of an unspecified distribution are discussed in Sen and Ghosh (1981), Ghosh and Mukhopadhyay (1978) with some extensions and refinements by Chow et. al. (1981, 1982). Throughout the rest of the paper, the terms involving the random variable  $[\sum_{i=1}^n \delta_i = 0]$  are left out without any further indication since  $P[\sum_{i=1}^n \delta_i = 0]$  goes to zero at an exponential rate and all our scale factors will be like  $n^{-a}$  with  $a > 0$ .

## 2. The Sequential Procedure

Let  $m(\geq 1)$  be the initial sample size. Then motivated by [1.4] define the stopping number  $N_c (\equiv N(c))$  by

$$N_c = \min\{n \geq m : n \geq (\frac{a}{c})^{1/\gamma} (\hat{\sigma}_n^2 + n^{-\gamma})\} \quad [2.1]$$

where  $\gamma > 0$  is a constant and  $\hat{\sigma}_n^2$  is an appropriate estimator of  $\sigma^2(\theta) \equiv \theta^2/E\delta = \theta^3(\int_0^\infty FG)^{-1}$ . Notice that by the ordinary Central Limit Theorem we have from [1.1]

$$n^{1/2}(\hat{\theta}_n - \theta) \rightarrow_D N(0, \sigma^2(\theta)). \quad [2.2]$$

An appropriate estimator of  $\sigma^2(\theta)$  based on  $\{(Z_i, \delta_i) : 1 \leq i \leq n\}$  is  $\hat{\sigma}_n^2$  where

$$\hat{\sigma}_n^2 = \hat{\theta}_n^3 / \bar{Z}_n = \{\bar{Z}_n^2 / \bar{\delta}_n^3\} [\bar{\delta}_n \neq 0] \quad [2.3]$$

with the overscore denoting the usual corresponding sample mean and  $[A]$  denoting the indicator of the set  $A$ . Observe that for each fixed  $\theta$ ,  $\hat{\sigma}_n^2$  converges a.s. to  $\sigma^2(\theta)$  as  $n \rightarrow \infty$ .

Our proposed sequential scheme will take  $N(c)$  observations and estimate  $\theta$  by  $\hat{\theta}_{N(c)}$  with resulting risk

$$R_c^* = a E(\hat{\theta}_{N(c)} - \theta)^2 + c E(N(c)). \quad [2.4]$$

The main result of this paper is

Theorem 1. (Risk Efficiency). For each  $\gamma > 0$ ,

$$R_c^* / R_c^0 \rightarrow 1 \text{ as } c \rightarrow 0.$$

Furthermore, we can easily demonstrate

Theorem 2. (Asymptotic Normality of  $\hat{\theta}_{N(c)}$ ). As  $c \rightarrow 0$ , for each  $\theta$   $\{\sqrt{N(c)} (\hat{\theta}_{N(c)} - \theta) / \hat{\sigma}_{N(c)}\}$  converges in distribution to a standard normal random variable.

The proofs of these theorems are outlined in the next section. We begin with a demonstration of the expansion [1.3] and a lemma concerning the rate of growth of  $N(c)$  as  $c \rightarrow 0$ .

### 3. Proofs.

We begin with a Lemma on left truncated inverse binomial moments.

Lemma 1. Let  $V$  be a binomial random variable with parameters  $n, p$ .

Then for any positive integers  $k, \ell$

$$E(V^{-k} | V \geq \ell) \leq K(np)^{-k}$$

where the constant  $K$  does not depend on  $n$  and  $p$ .

Proof. Write  $n^{[k]} = n(n-1)\dots(n-k+1)$  and  $n^{-[k]} = (n^{[k]})^{-1}$ . Then

$$\begin{aligned} & \sum_{x=0}^n \{(x+k)^{-[k]} n^{[x]} p^x (1-p)^{n-x} / x!\} \\ &= (n+k)^{-[k]} \sum_{x=0}^n \{(n+k)^{[x+k]} p^x (1-p)^{n-x} / (x+k)!\} \\ &\leq (n+k)^{-[k]} p^{-k} \end{aligned}$$

Therefore since  $(x+k)^{[k]} / x^k$  is decreasing with respect to  $x$ ,

$$\begin{aligned} E(V^{-k} | V \geq \ell) &\leq (\ell+k)^{[k]} \ell^{-k} E\{(V+k)^{-[k]} | V \geq \ell\} \\ &\leq (\ell+k)^{[k]} \ell^{-k} (n+k)^{-[k]} p^{-k} \\ &\leq K(np)^{-k}. \end{aligned}$$

We now turn to the crucial expansion given in [1.3]. To this end note that from [1.1] we can write

$$\hat{\theta}_n - \theta = U_n V_n \quad [3.1]$$

where

$$U_n = (\bar{Z}_n - \theta \bar{\delta}_n); V_n = \bar{\delta}_n^{-1} [\bar{\delta}_n \neq 0], \quad [3.2]$$

We then have

Lemma 2. For any  $p \geq 1$ ,  $E|U_n|^{2p} = O(n^{-p})$  and  $E|V_n|^p = O(1)$ .



Proof. Since  $\delta$  is either 0 or 1 and  $0 \leq Z \leq X$  a.s., all moments of  $\delta$  and  $Z$  are finite. From [3.2], we see that  $U_n$  is the average of  $n$  iidrv's with zero mean and variance  $\theta^2 E\delta (= \theta EZ)$ . Then by the Marcinkiewicz-Zygmund-Hölder inequality we obtain

$$E|U_n|^{2p} \leq k_1 n^{-p} E|Z - \theta\delta|^{2p} = O(n^{-p}).$$

(Here, and throughout this sequel  $k_j$  is a generic constant not depending on  $n$ ).

For the second part of this lemma we apply Lemma 1 to the binomial random variable  $n\bar{\delta}_n$ , from which the required statement follows immediately. This concludes the proof of Lemma 2.

Before we proceed further, we point out that [1.3] follows from the equality  $\hat{\theta}_n - \theta = (E\delta)^{-1} U_n \{1 - V_n(\bar{\delta}_n - E\delta)\}$  where  $U_n$  and  $V_n$  are defined by [3.2].

The next Lemma gives a convergence rate (as  $c \downarrow 0$ ) for the stopping number  $N_c$  of [2.1]. First note that  $N_c \geq (\frac{a}{c})^{1/2(1+\gamma)}$  a.s. so that  $N_c \uparrow \infty$  a.s. as  $c \downarrow 0$ . In the sequel  $\text{int}[x]$  denotes the greatest integer  $\leq x$  and  $k_1, k_2, \dots$  denote constants independent of  $c$ , but could depend on  $G$  and  $\theta$ .

Lemma 3. Let  $0 < \epsilon < 1$  be arbitrary. Then as  $c \downarrow 0$

$$P[N_c \leq \text{int}[n_c^0(1 - \epsilon)]] = O(c)$$

and

$$P[N_c > \text{int}[n_c^0(1 + \epsilon)]] = O(c).$$

} [3.3]

Proof. Write  $b = (\frac{a}{c})^{1/2}$  and  $n_{1c} = \text{int}[b^{1/(1+\gamma)}]$ ,  $n_{2c} = \text{int}[n_c^0(1 - \epsilon)]$ .

On the set  $[N_c \leq n_{2c}]$  we have that  $n \geq b\hat{\sigma}_n$  for some  $n \in \{n_{1c}, \dots, n_{2c}\}$ . Therefore

$$\begin{aligned}
P[N_c \leq n_{2c}] &\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} \{\hat{\sigma}_n^2 \leq b n^{-1}\}\right] \\
&\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} \{\hat{\sigma}_n^2 - \sigma^2 \leq b^{-2} n_{2c}^2 - \sigma^2\}\right], \text{ where } \sigma^2 = \sigma^2(\theta) = \theta^2/E\delta, \\
&\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} \{|\hat{\sigma}_n^2 - \sigma^2| \geq \sigma^2 \epsilon(2 - \epsilon)\}\right] \\
&= P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\hat{\sigma}_n^2 - \sigma^2| \geq \eta\right], \text{ for some } \eta > 0. \quad [3.4]
\end{aligned}$$

$$\text{Now } \hat{\sigma}_n^2 - \sigma^2 = a_n(\bar{Z}_n - EZ) - b_n(\bar{\delta}_n - E\delta) \quad [3.5]$$

where

$$\begin{aligned}
a_n &= (\bar{Z}_n + EZ)\bar{\delta}_n^{-3} [\bar{\delta}_n \neq 0] \text{ and} \\
b_n &= (EZ)^2(E\delta)^{-3}(\bar{\delta}_n^2 + \bar{\delta}_n E\delta + (E\delta)^2)\bar{\delta}_n^{-3} [\bar{\delta}_n \neq 0].
\end{aligned}$$

Since  $a_n \rightarrow 2\theta(E\delta)^{-2} = a_0$  a.s. and  $b_n \rightarrow 3\theta^2(E\delta)^{-2} = b_0$  a.s. as  $n \rightarrow \infty$ , we obtain from [3.5] that the righthand side of [3.4] may be bounded by

$$P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{Z}_n - EZ| > \eta\right] + P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{\delta}_n - E\delta| > \eta\right] \quad [3.6]$$

for an  $\eta > 0$ . Each of the two terms in [3.6] are handled in the same fashion. Observe that  $\{|\bar{Z}_n - EZ| : n_{1c} \leq n \leq n_{2c}\}$  is a reverse submartingale to which the Kolmogorov maximal inequality applies. Thus

$$P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{Z}_n - EZ| \geq \eta\right] \leq k_2 E|\bar{Z}_{n_{1c}} - EZ|^{4+4\gamma} = o(c),$$

where  $k_1$  is a constant depending only on  $\gamma$ . Thus combining [3.4] through [3.6] yields the first statement of Lemma 3. The second is proven in much the same way and so the proof of the lemma is terminated.

We may utilize the expansion in [3.5] to obtain the uniform integrability of  $\{N_c/n_c^0 : 0 < c \leq c_0\}$  for some  $c_0$  sufficiently small; from this  $E\{N_c/n_c^0\} \rightarrow 1$  as  $c \rightarrow 0$  will follow.

To this end note from [2.1] that

$$\hat{\sigma}_{N(c)}/\sigma \leq N_c/n_c^0 \leq \hat{\sigma}_{N(c)-1}/\sigma + (N(c) - 1)^{-\gamma}/\sigma + (b\sigma)^{-1}.$$

Then since  $N_c \uparrow \infty$  a.s. (as  $c \downarrow 0$ ), we obtain  $N_c/n_c^0 \rightarrow 1$  a.s.,

Furthermore

$$(N_c/n_c^0)^2 \leq 8\{(\hat{\sigma}_{N(c)-1}/\sigma)^2 + \sigma^{-2}(N(c) - 1)^{-2\gamma} + (b\sigma)^{-2}\}.$$

Thus in order to show  $E(\sup_{c \leq c_0} (N_c/n_c^0)^2) < \infty$  for some  $c_0$  sufficiently

small, we need only examine  $E(\sup_{n \geq m} |\hat{\sigma}_n^2/\sigma^2 - 1|)$  for sufficiently large  $m$ .

However, from [3.5] and the convergences  $a_n \rightarrow a_0$ ,  $b_n \rightarrow b_0$  a.s. it

suffices to establish the finiteness of  $E(\sup_{n \geq m} |\bar{Z}_n - EZ|)$  and

$E(\sup_{n \geq m} |\bar{\delta}_n - E\delta|)$ . The second is trivial, while the first is bounded by four times

$E|\bar{Z}_m - EZ|^2$  since  $\{|\bar{Z}_n - EZ| : n \geq m\}$  is a reverse submartingale.

Proof of Theorem 2. Recall [3.1] and [3.2]. From the central limit theorem

$n^{1/2}U_n \rightarrow_d N(0, \theta^2 E\delta)$ , for each  $\theta > 0$ . Also  $V_n \rightarrow (E\delta)^{-1}$  a.s. Hence

$n^{1/2}(\hat{\theta}_n - \theta) \rightarrow_d N(0, \sigma^2)$ . We have noted that  $\hat{\sigma}_n^2 \rightarrow \sigma^2$  a.s. (as  $n \rightarrow \infty$ )

and  $N_c \uparrow \infty$  as  $c \downarrow 0$ . Hence, by [3.8], [3.13], and [3.15] below, we get

$$\sqrt{N(c)} (\hat{\theta}_{N(c)} - \theta) / \hat{\sigma}_{N(c)} \rightarrow_d N(0, 1) \text{ as } c \downarrow 0.$$

Proof of Theorem 1. In view of the fact that  $E\{N_c/n_c^0\} \rightarrow 1$ , [1.4] and

[2.4] we must verify that

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E(\hat{\theta}_{N(c)} - \theta)^2 = 1. \quad [3.7]$$

Recall the expansion [3.3]. We first show

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \theta)^2 [N_c \leq n_{2c}]\} = 0. \quad [3.8]$$

Now

$$E\{(\hat{\theta}_{N(c)} - \theta)^2 [N_c \leq n_{2c}]\} \leq k_3 \{EU_{N(c)}^2 [N_c \leq n_{2c}] + EW_{N(c)}^2 [N_c \leq n_{2c}]\}, \quad [3.9]$$

where

$$W_n = U_n V_n (\bar{\delta}_n - E\delta)$$

Now applying the maximal inequality to the reverse submartingale

$\{|\bar{Z}_n - \theta \bar{\delta}_n| : n_{1c} \leq n \leq n_{2c}\}$  in the third step and Lemma 3 we have

$$\begin{aligned} E\{U_{N(c)}^2 [N_c \leq n_{2c}]\} &\leq E\left(\max_{n_{1c} \leq n \leq n_{2c}} U_n^2 [N_c \leq n_{2c}]\right) \\ &\leq E^{1/(1+\gamma)} \left(\max_{n_{1c} \leq n \leq n_{2c}} |U_n|^{2+2\gamma}\right) (P[N_c \leq n_{2c}])^{\gamma/(1+\gamma)} \\ &\leq k_4 E^{1/(1+\gamma)} |U_{n_{1c}}^{2+2\gamma}| O(c^{\gamma/(1+\gamma)}) \\ &= O(n_{1c}^{-1}) O(c^{\gamma/(1+\gamma)}) . \end{aligned}$$

Since  $n_{1c} \sim c^{-1/2(1+\gamma)}$  and  $(cn_c^0)^{-1} \sim c^{-1/2}$  we have that

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{U_{N(c)}^2 [N_c \leq n_{2c}]\} = 0 . \quad [3.10]$$

$$\text{Similarly } E\{W_{N(c)}^2 [N_c \leq n_{2c}]\} = \sum_{n=n_{1c}}^{n_{2c}} E\{W_n^2 [N_c = n]\}$$

$$\leq \left(\sum_{n=n_{1c}}^{n_{2c}} E W_n^{2p}\right)^{1/p} (P[N_c \leq n_{2c}])^{1-1/p} , \quad p > 1 . \quad [3.11]$$

$$\begin{aligned} \text{Now } E W_n^{2p} &= E\{U_n^{2p} V_n^{2p} (\bar{\delta}_n - E\delta)^{2p}\} \\ &\leq E^{1/3}\{U_n^{6p}\} E^{1/3}\{V_n^{6p}\} E^{1/3}(\bar{\delta}_n - E\delta)^{6p} \\ &= \{O(n^{-3p}) O(1) O(n^{-3p})\}^{1/3} = O(n^{-2p}) , \end{aligned}$$

Thus the rhs in [3.11] is  $O(n_{1c}^{-2+1/p} c^{1-1/p}) = O(c^h)$ , where

$h = \{2(1+\gamma)\}^{-1}(2 - 1/p) + (1 - 1/p) > 1/2$  whenever  $p > (2\gamma + 3)/(3 + \gamma)$ . So

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{W_{N(c)}^2 [N_c \leq n_{2c}]\} = 0 \quad [3.12]$$

Collecting our results [3.10] and [3.12] we obtain [3.8]. An entirely analogous argument using the second part of Lemma 3 will show

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{\hat{\theta}_{N(c)}^2 [N_c > n_{3c}]\} = 0 , \quad [3.13]$$

where  $n_{3c} = [n_c^0(1 + \epsilon)]$ . Therefore, in order to verify [3.7] we are

left with proving

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \theta)^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.14]$$

However, from [1.3]  $\lim_{c \rightarrow 0} (cn_c^0)^{-1} E(\hat{\theta}_{n_c^0} - \theta)^2 = 1$ , and further from the inequality

$$\begin{aligned} & E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})(\hat{\theta}_{n_c^0} - \theta) [n_{2c} < N_c \leq n_{3c}]\} \\ & \leq E^{\frac{1}{2}}\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} E^{\frac{1}{2}}(\hat{\theta}_{n_c^0} - \theta)^2, \end{aligned}$$

we see that it suffices to establish

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.]$$

Once again we utilize the expression [3.3] from which we get

$$\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0} = (E\delta)^{-1} \{U_{N(c)} - U_{n_c^0} - (W_{N(c)} - W_{n_c^0})\}. \quad [3.16]$$

Observe that

$$\begin{aligned} & E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \\ & \leq k_5 [E\{(U_{N(c)} - U_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \\ & \quad + E\{(W_{N(c)} - W_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\}] \end{aligned} \quad [3.17]$$

Now

$$\begin{aligned} & E\{(U_{N(c)} - U_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \\ & \leq \max [E\{\max_{n_{2c} < n \leq n_c^0} (U_n - U_{n_c^0})^2\}, E\{\max_{n_c^0 < n \leq n_{3c}} (U_n - U_{n_c^0})^2\}]. \end{aligned} \quad [3.18]$$

Since  $\{U_n : n_{2c} < n \leq n_c^0\}$  is a reverse martingale,  $\{(U_n - U_{n_c^0})^2 : n_{2c} < n \leq n_c^0\}$

is a reverse submartingale to which the maximal inequality will be applied.

This yields

$$\begin{aligned} E\left\{\max_{n_{2c} < n \leq n_c} (U_n - U_{n_c}^0)^2\right\} &\leq 4 E\{U_{n_{2c}} - U_{n_c}^0\}^2 \\ &= 4 \{EU_{n_{2c}}^2 - EU_{n_c}^0\} \end{aligned} \quad [3.19]$$

Now  $EU_n^2 = (0^2 E\delta)n^{-1}$ ,  $n_{2c} = \text{int}[n_c^0(1 - \epsilon)]$  and  $cn_c^0 = 0(1)$ . Therefore the rhs of [3.19] is of order  $(cn_c^0)\epsilon(1 - \epsilon)^{-1}0(1)$ . Hence, by choosing  $\epsilon$  arbitrarily small we have that

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\left\{\max_{n_{2c} < n \leq n_c} (U_n - U_{n_c}^0)^2\right\} = 0. \quad [3.20]$$

The second term in [3.18] is handled in an entirely analogous fashion and will yield the result paralleling [3.20]. So

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{(U_{N(c)} - U_{n_c}^0)^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.21]$$

Finally we must dispose of the second term in [3.17]. This is easily handled using the bound  $EW_n^2 = 0(n^{-2})$  and the simple inequalities

$$\begin{aligned} E\{(W_{N(c)} - W_{n_c}^0)^2 [n_{2c} < N_c \leq n_{3c}]\} \\ \leq 4 \{EW_{n_c}^2 + \sum_{n=n_{2c}}^{n_{3c}} EW_n^2\} = 0(n_c^{0-2}) + 0(n_c^{0-1})\epsilon(1 - \epsilon)^{-2}. \end{aligned}$$

Then on selecting  $\epsilon$  arbitrarily small we obtain

$$\lim_{c \rightarrow 0} (cn_c^0)^{-1} E\{(W_{N(c)} - W_{n_c}^0)^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.22]$$

Therefore in conjunction with [3.17], [3.21], and [3.22] yield [3.15].

This concludes the proof of Theorem 1.

#### 4. Concluding Remarks.

To start with, it is worth pointing out that the results are true even if  $G$  is degenerate at any point  $T$ . That is, the results hold even in the fixed point truncation of  $X_1$ . Although the results are stated here only when  $Y_1, \dots, Y_n, \dots$  are i.i.d., similar results hold even if  $Y_1, \dots, Y_n, \dots$  are assumed to be independent only. In the latter case however, some sort of condition like  $\lim_{n \rightarrow \infty} \int_0^\infty F \bar{G}_n > 0$ , with  $n \bar{G}_n = \sum_{i=1}^n G_i$ , needs to be imposed.

The final comment concerns the damping factor  $n^{-\gamma}$  introduced in the definition (2.1) of the stopping random variable  $N_c$ . Without this factor, the sequential estimator  $\hat{\theta}_{N_c}$  may not be asymptotically risk efficient as the following argument suggests. Define

$$N_{c,1} = \min\{n \geq m : n \geq (\frac{a}{c})^{\frac{1}{2}} \hat{\sigma}_n\} \quad [4.1]$$

where  $\hat{\sigma}_n$  is defined by [2.3]. With these definitions, it can be seen that if the censoring distribution  $G$  has positive mass at zero, then the risk  $R_{c,1}^*$  of the sequential procedure  $\hat{\theta}_{N_{c,1}}$  is at least  $\theta^2 p^m$  and  $R_{c,1}^*/R_c^0 \rightarrow \infty$  as  $c \rightarrow 0$  and hence  $\hat{\theta}_{N_{c,1}}$  is asymptotically totally risk inefficient. Thus the factor  $n^{-\gamma}$  in the definition of  $N_c$  appears to be important.

# REFERENCES

1. Chow, Y. S., and Yu, K. F. (1981). The performance of a sequential procedure for the estimation of the mean. Ann. Statist. 9, 184-189.
2. Chow, Y. S., and Martinsek, A. T. (1982). Bounded regret of a sequential procedure for estimation of the mean. Ann. Statist. 10, 909-914.
3. Gardiner, J. C., and Susarla, V. (1982). Sequential estimation of the mean survival time under random censorship. Research Memorandum, Dept. of Statistics & Probability, Michigan State University.
4. Ghosh, M. and Mukhopadhyay, N. (1978). Sequential point estimation of the mean when the distribution is unspecified. Commun. Statist. A8(7), 637-652.
5. Hannan, J. H. (1982). Private Communication.
6. Robbins, H. (1959). Sequential estimation of the mean of a normal population. Probability & Statistics (H. Cramér Volume), Almqvist & Wiksell, Uppsala, 235-245.
7. Sen, P. K., and Ghosh, M. (1981). Sequential point estimation of estimable parameters based on U-statistics. Sankhyā 43.
8. Starr, N. (1966). On the asymptotic efficiency of a sequential procedure for estimating the mean. Ann. Math. Statist. 37, 1173-1185.
9. Starr, N. and Woodroffe, M. (1969). Remarks on sequential point estimation. Proc. Nat. Acad. Science 61, 285-288.
10. Starr, N. and Woodroffe, M. (1972). Further remarks on sequential estimation: The exponential case. Ann. Math. Statist. 43, 1147-1154.



Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
District _____	
Date _____	
Post	Specs

A



REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
	AD A123913	
4. TITLE (and Subtitle)		5. TYPE OF REPORT & PERIOD COVERED
Risk-efficient estimation of the mean exponential survival time under random censoring		
7. AUTHOR(s)		6. PERFORMING ORG. REPORT NUMBER
Joseph C. Gardiner V. Susarla		
9. PERFORMING ORGANIZATION NAME AND ADDRESS		8. CONTRACT OR GRANT NUMBER(s)
Michigan State University East Lansing, MI 48824		N00014-79-C0522
11. CONTROLLING OFFICE NAME AND ADDRESS		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE
Statistics & Probability Program Office of Naval Research Arlington, VA 22217		January 1983
		13. NUMBER OF PAGES
		13
		15. SECURITY CLASS. (of this report)
		Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)		
APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
censored data, exponential distribution, and sequential estimation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		
<p>The paper proposes a sequential estimator <math>\hat{\theta}</math> of the parameter <math>\theta</math> of an exponential distribution when the data is censored. Without any further conditions, it is shown that <math>\hat{\theta}</math> is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of <math>\theta</math> plus a linear cost function of the number of observations. In addition, it is shown that <math>\hat{\theta}</math> is asymptotically normal as the cost per observation to zero.</p>		

DD FORM 1473

1 JAN 73

EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)